# Recap: Ch. 4

- Consider nonlinear sys.

$$\dot{x} = f(x)$$

- Step 1: look for eqlb. points

$$f(x) = 0$$

- Step 2: linearize around eqlb. point.

$$\dot{z} = A z$$

$$\text{where } A = \frac{\partial f}{\partial x}(x_e)$$

eqlb. point.

① if $Re(\lambda) < 0$ $\forall$ eigenvalues of $A$
then $x_e$ is AS
In fact, it is exp. stable.

② if $Re(\lambda) > 0$ for some $\lambda$     Thm. 4.7
then $x_e$ is unstable

what can't linearization answer?

- what if $Re(\lambda) = 0$ (e.g. $\dot{x} = -x^3$)
- No global conclusion
- No region of attraction!

Step 3: Lyapunov function method.

find $V$ s.t. $V(x) > 0 \ \forall x \neq 0$ and $V(0) = 0$

① if $\dot{V}(x) \leq 0 \ \forall x \in D \Rightarrow$ stable

② if $\dot{V}(x) < 0 \ \forall x \in D - \{0\} \Rightarrow$ AS

③ if $\dot{V}(x) < 0 \ \forall x \in \mathbb{R}^n - \{0\} \Rightarrow$ GAS

and $V(x) \to \infty$ as $\|x\| \to \infty$

Thm. 4.1
Thm. 4.2

- what if $\dot{V}(x) \leq 0$ (No strict inequality)

Step 4: LaSalle's invariance principle. Thm. 4.4

- If $\dot{V}(x) \leq 0$, let $E = \{ x \in \mathbb{R}^n ; \dot{V}(x) = 0 \}$

Then, all bounded solutions $x(t) \to \underbrace{M}_{\text{the largest invariant set in } E}$

Step 5: exponential Convergence thm 4.10

$k_1 \|x\|^2 \leq V(x) \leq k_2 \|x\|^2$ $\forall x \in D$ $\Rightarrow$ exp stable
and $\dot{V}(x) \leq -k_3 \|x\|^2$

If $D = \mathbb{R}^n$ $\Rightarrow$ globally exp stable

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What we did not cover?

- How to use Lyapunov function to show unstable (thm. 4.3)

- Time-varying systems $\dot{x} = f(t, x)$

$$V(t,x), \quad \dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$$

Section 4.5

- Converse Lyapunov thm.

existence of Lyapunov func for stable systems. Sec. 4.7

# Gradient flow:

- Motivation: optimization in Continuous-time
- Let $J : \mathbb{R}^n \to \mathbb{R}$ be $C^1$
- A point $x^*$ is global min of $J(x)$ if:

$$J(x^*) \leq J(x) \quad \forall x \in \mathbb{R}^n$$

- Necessary condition: $\nabla J(x^*) = 0$

where

$$\nabla J(x) = \begin{bmatrix} \frac{\partial J}{\partial x_1}(x) \\ \vdots \\ \frac{\partial J}{\partial x_n}(x) \end{bmatrix} \quad \text{is the gradient}$$

- gradient descent algorithm to find the min

$$x_{k+1} = x_k - \eta \nabla J(x_k) \quad , \quad k = 0, 1, 2, \ldots$$

- Continuous-time limit as $\eta \to 0$

$$\dot{x} = -\nabla J(x) \quad , \quad x(0) = x_0$$

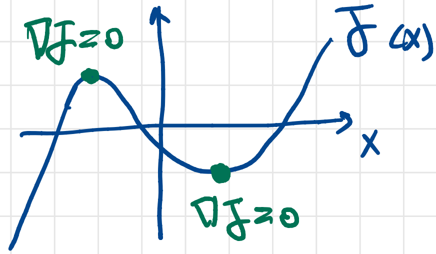
- Find the eqlb. points $\longrightarrow$ critical points.

$$E = \{x \in \mathbb{R}^n ; \nabla f(x) = 0\} \longrightarrow \text{all points where derivative is zero}$$

e.g. $f(x) = x^3 - x$

$$\nabla f(x) = 3x^2 - 1 = 0$$

$$\Longrightarrow x = \pm\sqrt{\tfrac{1}{3}}$$




- Linearize around eqlb. points. $x_e$

$$A = -\frac{\partial}{\partial x}(\nabla f)_{(x_e)} = -\nabla^2 f_{(x_e)} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x_e)\right]$$

- If $Re(\lambda) > 0$ $\forall$ all eigenvalues of $A$ $\Longrightarrow$ exp stable

$$\Updownarrow$$

$$A = \nabla^2 f(x_e) \text{ is p.d. matrix}$$

$$\Updownarrow$$

$x_e$ is a local minimizer

Example: $f(x) = x^3 - x$

$\nabla f(x) = 3x^2 - 1 \Rightarrow x_e^{(1)} = +\sqrt{\frac{1}{3}}, \; x_e^{(2)} = -\sqrt{\frac{1}{3}}$

$\nabla^2 f(x) = 6x \Rightarrow$

$$\nabla^2 f(x_e^{(1)}) > 0 \xRightarrow{\text{local min}} \text{exp stable}$$

$$\nabla^2 f(x_e^{(2)}) < 0 \xRightarrow{\text{local max}} \text{unstable}$$

Lyapunov function:

$$V(x) = f(x) - f^* \geq 0$$

$$\Rightarrow \dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = -\nabla f(x)^T \nabla f(x)$$

$$= -\|\nabla f(x)\|_2^2 \leq 0$$

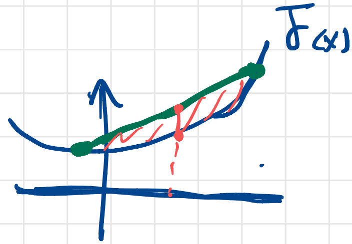
$\Rightarrow$ By LaSalle (thm. 4.4.)

any bounded sol. $x(t) \to E = \{x \in \mathbb{R}^n : \nabla f(x) = 0\}$

the largest invariant set in $E$ is $E$

$\Rightarrow$ $x(t)$ converges to a critical point

- We need to assume more on $J(x)$ in order to conclude convergence to $x^*$




Convex functions:

- $J : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$J(\lambda x + (1-\lambda) y) \leq \lambda J(x) + (1-\lambda) J(y) \qquad \forall x, y \quad \forall \lambda \in [0,1]$$

- If $J \in C^1$, this is equivalent to

$$J(y) \geq J(x) + \nabla J(x)^T (y - x) \qquad \forall x, y$$

- For convex functions,

$$x^* \text{ is global min} \iff \nabla J(x^*) = 0$$

proof:

- ($\Longrightarrow$) is a necessary condition for optimality
- ($\Longleftarrow$) Suppose $\nabla J(x) = 0 \Longrightarrow$

$$J(y) \geq J(x) + \underbrace{\nabla J(x)}_{0}{}^T (y - x) = J(x) \qquad \forall y$$

- Assume $J$ is convex and try Lyapunov func.

$$V(x) = \frac{1}{2}\|x - x^*\|^2$$

$$\Rightarrow \dot{V}(x) = -(x - x^*)^T \nabla J(x)$$

$$= (x^* - x)^T \nabla J(x)$$

- By convexity, $J(x^*) \geq J(x) + \nabla J(x)^T (x^* - x)$

$$\Rightarrow \dot{V}(x) \leq -\left(J(x) - J(x^*)\right) \leq 0$$

- $\dot{V}(x) = 0 \Rightarrow J(x) = J(x^*)$

$\Rightarrow$ $x$ is also a global min.

$E = \{x \in \mathbb{R}^n ; \dot{V}(x) = 0\} = \{$ all global minimizers $\}$

$E$ is largest invariant set because

$\dot{x} = \nabla J(x) = 0$ if $x$ is global min

$\Rightarrow x(t) \longrightarrow$ a global minimizer.

- Can we have a rate of convergence?

$$\dot{V}(x) \leq -(J(x) - J(x^*))$$

integrate with time

$$\Rightarrow V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(s))\, ds \leq -\int_0^t J(x(s)) - J(x^*)\, ds$$

$$\Rightarrow \frac{1}{t}\int_0^t J(x(s))\, ds - J(x^*) \leq \frac{1}{t}V(x(0)) - \frac{1}{t}V(x(t))$$

$$\leq \frac{1}{t}V(x(0)) = \frac{1}{2t}\|x_0 - x^*\|^2$$

Because $J(x(s))$ is decreasing

$$\Longrightarrow$$

$$J(x(t)) - J(x^*) \leq \frac{1}{t}\int_0^t J(x(s))\, ds - J(x^*) \leq \frac{1}{2t}\|x_0 - x^*\|^2$$

$$\Rightarrow \boxed{J(x(t)) - J(x^*) \leq \frac{1}{2t}\|x_0 - x^*\|^2}$$

- In order to have exp convergence, we need stronger assumptions.

**Strong Convexity:** $\exists\, \alpha > 0$ s.t.

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\alpha}{2}\|y - x\|^2 \quad \forall x, y$$

Let $\quad V(x) = \frac{1}{2}\|x - x^*\|^2$

$$\Rightarrow \quad \dot{V}(x) = -(x - x^*)^T \nabla f(x)$$

$$\leq -(f(x) - f(x^*)) - \frac{\alpha}{2}\|x - x^*\|^2$$

$$\leq -\alpha V(x)$$

Comparison lemma

$$\Rightarrow \quad V(x(t)) \leq e^{-\alpha t}\, V(x(0))$$

$$\Rightarrow \quad \underbrace{\|x(t) - x^*\|^2 \leq e^{-\alpha t}\, \|x(0) - x^*\|^2}_{\text{exponential convergence}}$$